Problem Set 1 (Due 1/15/20 11:59PM)

Your Name:

Names of Any Collaborators:

Please review the problem set guidelines in the syllabus before turning in your work. Additionally:

(a) you may use freely any results that we discussed during the lecture;

(b) you may use any results that are proved in Chapter 1 of the textbook;

(c) you may use any exercises from Chapter 1 of the textbook if and only if they do not render the problem trivial.

In any case, you should make it clear when a statement in your proof follows from another result, and which result you are using.

P3 (a) Let $z \in \mathbb{C}$. Using the principle of mathematical induction, show that the formula holds for all $n \in \mathbb{N}$:

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}.$$

(b) If $\omega_1, \ldots, \omega_n$ are the *distinct* n-th roots of unity, show that $\sum_{i=1}^{n} \omega_i = 0$.

(c) Compute the following sum of real numbers:

$$\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} + \cos\frac{7\pi}{7} + \cos\frac{9\pi}{7} + \cos\frac{11\pi}{7} + \cos\frac{13\pi}{7} + \cos\frac{15\pi}{7}.$$

You must justify your computation.

Proof. (a) We argue by induction. The basic step: if $n = 1$, then $1 + z = \frac{(1+z)(1-z)}{1-z} = \frac{1-z^2}{1-z}$. Let $n \in \mathbb{N}$ and assume (for the inductive hypothesis) that the equation holds:

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}.$$

Then we have

$$\begin{aligned} 1 + z + z^2 + \cdots + z^n + z^{n+1} &= 1 + z\left(1 + z + z^2 + \cdots + z^n\right) \\ &= 1 + z\left(\frac{1 - z^{n+1}}{1 - z}\right) \\ &= \frac{1 - z}{1 - z} + \frac{z - z^{(n+1)+1}}{1 - z} \\ &= \frac{1 - z^{(n+1)+1}}{1 - z}. \end{aligned}$$

By the principal of mathematical induction, the claim is proved.

(b) As seen in the lecture, there are n distinct n-th roots of unity, given by

$$1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}$$

where $\omega_n = e^{i\frac{2k\pi}{n}}$.[1] The point is that the distinct roots can be written as powers of the ω_n and hence we can take $z = \omega_n$ in part (a), to obtain

$$\begin{aligned} 1 + \omega_n + \omega_n^2 + \cdots + \omega_n^{n-1} &= \frac{1 - \omega_n^{(n-1)+1}}{1 - \omega_n} \\ &= \frac{1 - \omega_n^n}{1 - \omega_n} \\ &= \frac{1-1}{1 - \omega_n} && \text{(Recall that } \omega_n^n = 1\text{)} \\ &= 0. \end{aligned}$$

(c) We can interpret the real number $\cos \frac{(2k+1)\pi}{7}$ as the real part of a complex number using Euler's formula. In fact,

$$\cos \frac{(2k+1)\pi}{7} = \operatorname{Re}\left(\frac{(2k+1)\pi}{7} + i \sin \frac{(2k+1)\pi}{7}\right) = \operatorname{Re}\left(e^{i(2k+1)\pi/7}\right).$$

Define $c_0 = e^{i\pi/7}$ and write $\omega_7 = e^{i2\pi/7}$ (a 7-th root of unity, as in the proof of (b)). Notice that

$$e^{i(2k+1)\pi/7} = e^{i\pi/7} e^{i2k\pi/7} = c_0 \omega_7^k.$$

Putting this together, we obtain

$$\begin{aligned} \sum_{k=0}^{7} \cos \frac{(2k+1)\pi}{7} &= \sum_{k=0}^{7} \operatorname{Re}\left(e^{i(2k+1)\pi/7}\right) \\ &= \operatorname{Re}\left(\sum_{k=0}^{7} c_0 \omega_7^k\right) && \text{(Re is additive)} \\ &= \operatorname{Re}\left(c_0 \sum_{k=0}^{6} \omega_7^k + c_0 \omega_7^7\right) \\ &= \operatorname{Re}\left(c_0 \cdot 0 + c_0 \cdot 1\right) && \text{(by part (b) \& } \omega_7 \text{ is a 7-th root of unity.)} \\ &= \operatorname{Re}\left(e^{i\pi/7}\right) \\ &= \cos \pi/7 \end{aligned}$$

Note also that $\cos \pi/7 = \cos 15\pi/7$ so either answer is correct.

P4 Let $z, w \in \mathbb{C}$.

(a) Prove the formula

$$|z+w|^2 = |z|^2 + 2 \operatorname{Re} z\overline{w} + |w|^2.\text{[2]} \tag{1}$$

[1] Note: the notation might be confusion, the ω_n given here is different that the ω_n in the problem statement, which could have been any one of the roots.

[2] Notice that $\operatorname{Re} z\overline{w} = \operatorname{Re} w\overline{z}$.

(b) From (1), deduce the parallelogram law

$$|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2$$

Give a geometric interpretation of this formula.

Proof. (a) We have

$$\begin{aligned}
|z+w|^2 &= (z+w)(\overline{z+w}) \\
&= (z+w)(\overline{z}+\overline{w}) \\
&= z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} \\
&= |z|^2 + z\overline{w} + w\overline{z} + |w|^2.
\end{aligned}$$

Now it suffices to show that $z\overline{w} + w\overline{z} = 2\operatorname{Re} z\overline{w}$. write $z = x+iy$ and $w = a+ib$. We have

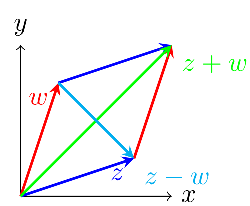
$$\begin{aligned}
z\overline{w} + w\overline{z} &= (x+iy)(a-ib) + (a+ib)(x-iy) \\
&= (xa+yb) + i(-xb+ya) + (ax+by) + i(-ay+bx) \\
&= 2(ax+by) \\
&= 2\operatorname{Re} z\overline{w}.
\end{aligned}$$

This proves (a).

(b) Applying (a), we obtain

$$\begin{aligned}
|z+w|^2 + |z-w|^2 &= |z|^2 + 2\operatorname{Re} z\overline{w} + |w|^2 + |z|^2 + 2\operatorname{Re} z(\overline{-w}) + |-w|^2 \\
&= |z|^2 + 2\operatorname{Re} z\overline{w} + |w|^2 + |z|^2 - 2\operatorname{Re} z\overline{w} + |w|^2 \\
&= 2|z|^2 + 2|w|^2.
\end{aligned}$$

Interpreting the complex numbers z, w geometrically (as vectors), the parallelogram law says that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the lengths of its sides. See the figure:



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P6 (a) Find a necessary and sufficient condition on $z \in \mathbb{C}$ such that $|z-i| = |z+i|$.

(b) Repeat with the condition $|z-1| = |z+1|$.

Solution. (a) Geometrically, the points satisfying the equation $|z-i| = |z+i|$ are points equidistant from both i and $-i$, i.e., the real line. So we claim that z is real if and only if $|z-i| = |z+i|$. To prove this, first suppose $z = x \in \mathbb{R}$. Then $|z-i|^2 = x^2 + (-1)^2 = x^2 + 1^2 = |z+i|^2$. Conversely, suppose $|z-i| = |z+i|$. Writing $z = x+iy$, we have

$$\begin{aligned}(y-1)^2 &= x^2 + (y-1)^2 - x^2\\ &= |x+iy-i|^2 - x^2\\ &= |x+iy+i|^2 - x^2\\ &= x^2 + (y+1)^2 - x^2\\ &= (y+1)^2.\end{aligned}$$

The only solution to the equation $(y-1)^2 = (y+1)^2$ is $y = 0$. Hence, $\operatorname{Im} z = y = 0$ so that z is real.

(b) Note: the solution is basically the same as part (a). Geometrically, the points satisfying this equation are equidistance from 1 and -1, i.e., the imaginary line. So we claim that z is purely imaginary if and only if $|z-1| = |z+1|$. If $z = iy$, then $|z-1|^2 = |iy-1|^2 = (-1)^2 + y^2 = 1^2 + y^2 = |iy+1|^2 = |z+1|^2$. Conversely, suppose $|z-1| = |z+1|$ and write $z = x+iy$. Then

$$\begin{aligned}(x-1)^2 &= (x-1)^2 + y^2 - y^2\\ &= |x+iy-1|^2 - y^2\\ &= |x+iy+1|^2 - y^2\\ &= (x+1)^2 + y^2 - y^2\\ &= (x+1)^2.\end{aligned}$$

This time we reach the conclusion that $x = 0$ so that $\operatorname{Im} z = x = 0$ and z is purely imaginary.

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P9 Let $\varepsilon > 0$. Prove that the open disk $D_\varepsilon(z_0) \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z-z_0| < \varepsilon\}$ is a domain.

Proof. The disk is *nonempty* since $|z_0 - z_0| = 0 < \varepsilon$ so $z_0 \in D_\varepsilon(z_0)$. To prove that it is *open*, we will show that every point is interior. So let $w_0 \in D_\varepsilon(z_0)$. By definition, this means that $|w_0 - z_0| < \varepsilon$. Define $\varepsilon' = \varepsilon - |w_0 - z_0|$. Notice that $\varepsilon' > 0$ since $|w_0 - z_0| < \varepsilon$. Therefore, we claim that $D_{e'}(w_0) \subseteq D_\varepsilon(z_0)$. To prove this, let $w \in D_{\varepsilon'}(w_0)$ so that $|w - w_0| < \varepsilon'$. To show that $w \in D_\varepsilon(z_0)$, we compute

$$\begin{aligned}|w-z_0| &= |w-w_0+w_0-z_0|\\ &\stackrel{T.I.}{\leq} |w-w_0| + |w_0-z_0|\\ &< \varepsilon' + |w_0-z_0|\\ &= \varepsilon.\end{aligned}$$

This proves that $w \in D_\varepsilon(z_0)$ so that $D_{e'}(w_0) \subseteq D_\varepsilon(z_0)$. We just proved that every point is interior and hence $D_\varepsilon(z_0)$ is an open set. See Figure 1 to see the geometry.

Finally, to show that $D_\varepsilon(z_0)$ is *connected*, let z_1, z_2 be any distinct points in $D_\varepsilon(z_0)$. The line segment joining z_0 to z_1 is contained in the disk. Similarly, the line segment joining z_0 to z_2 is contained in the disk. You could parameterize these line segments as $r_i(t) = z_0 + t(z_i - z_0)$,

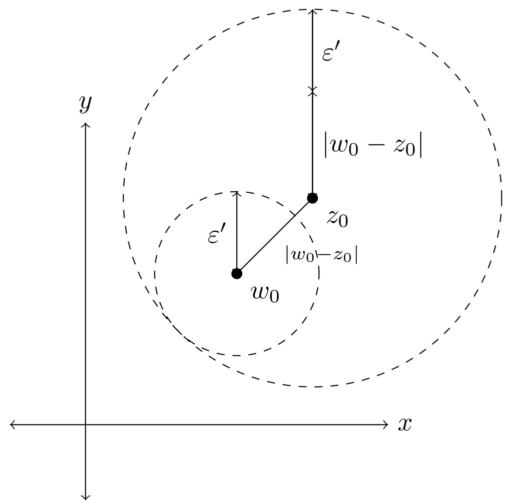

Figure 1: The proof that the disk is open.

$t \in [0,1]$ for $i = 1, 2$. If $w = r_i(t)$ is a point on such a line segment, then it belongs to $D_\varepsilon(z_0)$ since

$$\begin{aligned} |w - z_0| &= |z_0 + t(z_i - z_0) - z_0| \\ &= |t(z_i - z_0)| \\ &= |t||z_i - z_0| \\ &< 1 \cdot \varepsilon && (0 \le t \le 1 \text{ and } |z_i - z_0| < \varepsilon \text{ since } z_i \in D_\varepsilon(z_0)). \end{aligned}$$

Joining these end to end, we obtain a polygonal line connecting z_1 to z_2 that lies totally in the disk. Alternatively, you could take the straight line segment between z_1 and z_2, but it is slightly harder to say why this lies in the set.

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